Exact Differential Equation Definition

A first-order differential equation $M(x,y)dx + N(x,y)dy = 0$ is an exact differential equation iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example-1: The differential equation $(xy^2 + x)dx + yx^2dy = 0$ is exact because $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(xy^2 + x) = 2xy$ and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(yx^2) = 2xy$ are equal.	**Example-2:** The differential equation $(y^2 + 1)dx + xydy = 0$ is not exact because $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y^2 + 1) = 2y$ and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(xy) = y$ are not equal.
Example-3: The differential equation $\cos y\, dx + (y^2 - x\sin y)\, dy = 0$ is exact because $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(\cos y) = -\sin y$ and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(y^2 - x\sin y) = -\sin y$ are equal.	**Example-4:** The differential equation $\cos y\, dx + (y^2 + x\sin y)\, dy = 0$ is not exact because $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(\cos y) = -\sin y$ and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(y^2 + x\sin y) = \sin y$ are not equal.

Solution of exact DE

For the solution we have

(i) integrate M w. r. to x regarding y as constant
(ii) Find out those terms in N which are free from x and integrate them w.r.to y
(iii) Add the two expressions so obtained and equate the sum to an arbitrary constant

This will give the general solution of the given exact DE.

The solution is $\int M dx + \int_{\substack{\text{terms free}\\ \text{from x}}} N dy = C$

Solving an Exact Differential Equation

Solve the differential equation $(2xy - 3x^2)dx + (x^2 - 2y)dy = 0$ Solution: Here $M = (2xy - 3x^2)$ and $N = (x^2 - 2y)$ Then $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy - 3x^2) = 2x$ and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2 - 2y) = 2x$ are equal.	The solution is $\int M dx + \int_{\substack{\text{terms free}\\ \text{from x}}} N dy = C$ $\Rightarrow \int (2xy - 3x^2)\, dx + \int_{\substack{\text{terms free}\\ \text{from x}}} (-2y)dy = C$ $\Rightarrow \left[2y\frac{x^2}{2} - 3\frac{x^3}{3}\right] - \frac{2y^2}{2} = C$ $\Rightarrow x^2y - x^3 - y^2 = C$

Exercise:

Determine whether the following differential equations is exacts if it is find its solution.

(i) $(y^4 + 4x^3y + 3x)dx + (x^4 + 4xy^3 + y + 1)dy = 0$
(ii) $(x - 2e^{-y})dy + (y + x\sin x)dx = 0$
(iii) $(2x^3 + 3y)dx + (3x + y - 1)dy = 0$
(iv) $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$